**Vector Spaces and Subspaces**

Table of Contents

[Vector Subspaces 3](#_Toc66212124)

[Column Spaces 4](#_Toc66212125)

[Null Spaces 7](#_Toc66212126)

[Finding the Null Space 8](#_Toc66212127)

[Row Reduced Form 10](#_Toc66212128)

[Complete Solution for 11](#_Toc66212129)

[Independence 16](#_Toc66212130)

[Basis of a Space 17](#_Toc66212131)

[Rank One Matrices 20](#_Toc66212132)

[Graphs 21](#_Toc66212133)

( not squared) is a vector space that has all the column vectors with 2 real components. It is essentially the plane. For any vector , all values of and will result in a vector that is within that vector space, be it , or even . Similarly, is the vector space that has all columns vectors with 3 real components and is the vector space that has all the column vectors with real components.

If we remove the vector from any vector space, the vector space is no longer a vector space. The reason for this is that all vectors start at the origin. If the origin is not present, none of the other vectors can be present either. This get a little more complicated if we consider what happens when we remove some other vector, and we will consider that a little later, when discussing column spaces.

## Vector Subspaces

First, we need to understand what a vector space is. A vector space is any space containing vectors such that if any of the vectors are multiplied with a scalar value or any two vectors are added, the resulting vector remains within the space. Consider just the first quadrant of the plane. If we multiply a vector in that quadrant with , we end up in the third quadrant. Thus, just the first quadrant is not a vector space.

For , there are three possible vector spaces inside . The origin itself could be considered a vector space since it meets both the conditions. Any line through the origin is also a vector space. It must be through the origin since otherwise, if we multiplied a vector by , we would leave the vector space. Finally, itself is a vector space. Each of these three is a subset of . They are called the vector subspaces of . Similarly, the vector subspaces of are the origin, any line through the origin, any plane through the origin and itself.

## Column Spaces

The matrix could be divided into two column vectors, and . The vector subspace created by these column vectors is what we are interested in. The two vectors on their own cannot form a vector space, since there are linear combinations that will lie outside of the space. Thus, all linear combinations of the two vectors makes up the vector space needed. We already know that all linear combinations of two column vectors makes a plane in 3D space. Thus, the column space of is a plane in containing the two column vectors and .

Now look back at the question of what happens if we remove a single vector, that is not the origin, from any vector space. With all the other vectors present, there would be some linear combination that would give us the vector we just removed. Thus, if we remove any vector at all from a vector space, it ceases to be a vector space.

Expanding on the idea of column spaces, consider the matrix . We can form linear combinations with the two column vectors in to reach any point on the plane, so the column space of is . For the matrix , we can use linear combinations only to create vectors that are multiples of the column vector , since is linearly dependant. This means the column space is a line through the origin. With , we can use linear combinations to reach any point of the plane, so the column space is .

Questions:

For the column vectors and

1. If is a vector subspace created by and is a vector subspace created by , what is ?

For questions like this, it is best to try and draw a diagram of the vectors. The two vectors are linearly independent, so their intersection point is the origin.

2. a) Find a subspace .

is a plane containing the vectors and .

b) Could be equal to ?

No. The union of the two vectors is just two lines, while is an entire plane. Linear combinations line , which are included in , are not included in .

c) Find a subspace of such that is not included in or .

can be any line through the origin other than , or any scalar multiplied by or . We must ignore the origin even though it is present in every vector space, since a vector space is not possible without the origin.

d) Find the intersection of with the plane.

The intersection of any two planes must be a line. In this case, the line must be or some scalar multiple of . The reason for this is, in the plane, the -coordinate is , which is also true for , meaning lies on the plane.

If is a plane through the origin in and is a line through the origin in , is a vector space?

It is a vector space only if lies on . Otherwise, their linear combinations will not lie within the space, so it will not be a vector space.

Consider the following system :

Can this be solved for any value of ?

If , solutions include and . If however, there is no solution.

Essentially, has to be a linear combination of the columns of to be solvable. This means must be in the column space of . Also, notice that column 3 = column 1 + column 2, thus the columns are not linearly independent. We can remove any one of the columns without changing the column space.

## Null Spaces

For a system , the null space of is all the solutions of for which . For the system given earlier, such solutions include , , , , etc. Essentially, all scalar multiples of the vector , which is , are solutions. Thus, all the solutions create a line through the origin.

If we take and as any other vector in this set, then and . Thus, this is a subspace. The solutions of create a vector subspace which is called the null space of .

The reason null spaces are special is because they are unique. Consider the system where . The origin, , cannot be a solution to this. Since the origin is not a solution, the set of solutions to the system can never create a vector subspace of its own. The null space is the only possible vector subspace.

For a matrix in which the columns are linearly independent, no linear combination will give as a solution other than itself. Thus, the null space contains only the origin.

## Finding the Null Space

We can find the null space for a matrix. Say for some system , the matrix is

If we attempt to perform Gaussian elimination on this matrix, we end up with this:

There is no way to get all the pivots for this matrix, since the columns are not linearly independent. This is known as the Row Echelon Form.

We have two pivots here, from column 1 and from column 3. (Remember that a pivot is just the non-zero value that performs elimination and has nothing to do with the diagonal.) These two columns are called the pivot columns. The other two columns are called the free columns.

The row echelon form gives us two equations:

and

Notice that the free columns give us the terms in the first equation and in the second equation. We can take any values for these two terms, but commonly we take one as and the others as . Say and .

This is one solution to the system. Actually, should be written as all scalar multiples of the column vector are also solutions.

Next, if we consider and ,

For any system, to find the null space, we need to find such solutions, where is the number of columns and is the ‘rank’ of the matrix, which is the number of pivots it has. For our case, and , so we need solutions, which we already have.

Thus, is the null space for .

For a matrix with pivots, we would need solutions.

Notice that the number of special solutions we need is the same as the number of free columns we have.

## Row Reduced Form

From the last lecture, recall that we had a matrix

If we perform upwards elimination on this matrix, we get

Pulling the two pivot columns to one side and the two free columns to another we get

or

This final form is known as the Row Reduced Form or R-Matrix of the original matrix. This is just a form of representation for the matrix.

In MATLAB, there is a library function that gives us the R-Matrix for any input matrix.

## Complete Solution for

Recall that we mentioned that for any system where , the solutions cannot form a subspace since the origin cannot be included. However, there can be a number of solutions for any value of . We will be looking into how to find these solutions here.

Consider that . Thus, for any matrix we can form an augmented matrix:

Performing Gaussian elimination,

Notice that if any row in the matrix is a combination of any other rows, then we end up with a row of zeroes in the row echelon form.

An important observation to make is that if the matrix is such that , then the matrix is not solvable. That particular does not lie within the column space of .

Say . Then we have

This gives us two equations:

We can now begin to find the complete solution for the system.

First, we set the free columns to zero. Thus, and . From here we get a single solution, which we call . Here, .

Recall that, while trying to find the null space, we set the free columns to any values. By convention, one column was set to 1 while the rest was set to 0. This is in stark contrast to what we do here.

Next, we use the null space to find the solution. We know that the null space for the matrix is the linear combination of two vectors which are guaranteed to give us . If we add these to the value of we just found, we will always end up with the required solution. We already know the null space for the vector from the previous lecture, so the complete solution is given by:

We can plug in any value of and into this equation, but we will always end up with a solution to the system .

If the columns of matrix are not linearly dependent, then would cover the entirety of the space in . Thus, any value of would be solvable.

Consider the matrix . The row reduced form of this matrix is , which we have previously seen we can write as . We will be analysing this format now.

For matrices where the number of equations and unknows is the same, i.e. the row (m) and column (n) numbers are the same, we have a square matrix. Such a matrix will give us a pivot in every column, so the rank (r) is equal to n and m. Since we have free columns, we have special solutions, meaning we only have the origin in the null space of that matrix. Under this circumstance, the complete solution of the system is . The row reduced form of this matrix will be the identity matrix.

For matrices where , meaning we have more columns than rows, we get a rectangular matrix.

Consider . This equation can have infinitely many solutions.

For a rectangular matrix, we will have a pivot in every row (thus ), but not in every column. Since we have free columns, , and a null space exists so we have infinitely many solutions.

For matrices where , where we have more equations than columns, we get a pivot in every column, but not in every row.

Consider . Here, .

We get no free columns, so there is no null space other than the origin. Thus, we have a single solution. However, there is a twist. We need to take into consideration the solvability conditions. In the above example, the solution of the third row has to be . If we try to solve for a matrix which does not meet this condition, we will not get a solution.

So, in this scenario, we may not always have a solution, but if we do, then we will have a single solution.

Finally let us go back to the first matrix we say, where and . We get both free columsna and empty rows under this condition. Thus, . Since we have free columns, we have a null space, which means we have infinitely many solutions. Since we have a row, we will not always have a solution. Thus, we may not always have a solution, but if we do, then we will have infinitely many solutions.

For the following matrix, solve for .

The augmented matrix is:

and the row reduced form is:

If we attempt to find a particular solution to this system, we will see that , and . Thus, .

To find the null space, , we set the free columns to some value, say . Thus, , , so and . .

Thus, . Using different values of , we can find infinitely many solutions in . However, notice that we had to abide by the condition for this to be true.

## Independence

Formally, columns of a matrix are independent if they have no non-zero solution for . This means that the only solution is the origin, and there are no free columns.

If some combination other than gives a solution for , then there are columns that are dependent.

## Basis of a Space

Consider the matrix . Their span, or columns space, is the entirety of . However, we could cover this span using any two of the three columns. Say we choose and . These would thus form the minimum number of column vectors required to cover the span. They are called the basis of the space.

In , with a matrix, column vectors are required to form the basis of the space. The space will only be covered if the matrix is invertible, i.e. there are no non-zero solutions to .

We previously defined and as the basis for . Actually, any column vectors can form the basis, given they abide by two conditions, first, that the vectors are independent, and second, that their combination spans the entire space.

For example, the column vectors , and would form the basis for a plane in .

To span , we need exactly 3 vectors. If we used 4, we would still be able to span it, but we would have one dependent vector that we do not need. If we used 2, we would be unable to span the entirety of . The minimum number of vectors we need to form the basis that will allow us to span a certain space is called the dimension of that space.

Consider the matrix . Two of the columns are not independent. Thus, the span of the matrix is a plane in , the basis of the column space is any combination of two columns, except the first and fourth, since they are the same and would span a line instead of a plane, and the columns space has 2 dimensions.

We could also say that the vectors and form the basis since these are multiples of the first and third columns. This proves that the basis is not unique.

Notice that the dimensions for is equal to the rank (number of pivots) of . The dimensions of the null space of , , would be equal to here, the number of free columns. The basis for the null space would be the special solutions.

Consider another matrix, . First, find the row reduced echelon form. Finding this form will help us avoid mistakes.

From here, we can tell the rank is and that there are free columns. Thus, the matrix has dimensions = rank = . The span is (this form is used to represent ), and the basis is any one of the vectors, or some multiple of one of them. Say the basis is .

For the null space, the dimensions would be , the span would be the special solutions (this could be ).

If we did not have a null space, then the dimension of the null space would be , and it would have no span or basis.

Now consider the transpose of the matrix . The row reduced echelon form is . Thus, for the columns space of , , the dimension is , the span is and the basis is . Notice that the row from is the basis for . We could say that the columns space dimension of is the row space dimension of .

Now consider the null space of , . There, the dimension would be , the basis could be and the span could be .

We could say that . Thus,

Thus,

Thus, is called the left null space of , or the null space of the row space.

## Rank One Matrices

For matrices such as , we can only have one pivot. Thus, they are called rank one matrices. For such matrices, we can write them as a product of the first column and the first row, i.e. . If we have two column vectors and , then will give us a rank one matrix since we are multiplying a column by a row (if is a column, then is a row).

Rank one matrices are the building blocks for all other matrices.

## Graphs

Consider the following graph:

A

B

C

D

E

Graphs like this are used in all sorts of programs. They are used to store relations between webpages and on social media sites. For example, in this graph, and have a distance of between them. We could think of this as knows someone (), who knows . Thus, might know as well. This would cause a friend suggestion to pop up.

If we want to use graphs in linear algebra, we must first convert them into matrices. Consider this graph:

1

2

3

4

This is a directed graph. We can create an incidence matrix from this graph. In the matrix, the columns represent vertices and the rows represent edges. So we have 4 columns and 5 rows. We use positive values for incoming edges and negative values for outgoing edges. Thus,

We have a single edge between and , going out from and going in to . Thus, the first row has for the first position, for the second and s for the third and fourth since they are not involved. The rest of the rows work similarly.

Notice in the graph that we can reach from in two ways. Thus, it should be evident that . From the matrix, row 1 (connection of and ) + row 2 (connection of and ) = row 3 (connection of and ). Similarly, row 3 + row 4 = row 5. Whenever we have dependant rows in the incidence matrix, they indicate the existence of loops in the graph.

Also notice that the matrix follows a certain structure. In every row, we have 2 s, a and a .

Now consider the solution for for the incidence matrix.

Thus, we have , , , and . If we consider our original graph contains vertices at different potentials, with arrows pointing to where current is flowing, then the five equations we just got should represent the potential differences between the vertices. Thus, the matrix computes the differences between the vertices from to across each of the five edges.

Now, when is the potential difference between every vertex ? Of course, this can only happen when each vertex has the same value. Thus, without knowing anything at all about the values of , we can say that the solution to is .